Magma Yield Stress and Permeability: Insights From Multiphase Percolation Theory

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Abstract

Magmas often contain multiple interacting phases of embedded solid and gas inclusions. Multiphase percolation theory provides a means of modeling assemblies of these different classes of magmatic inclusions in a simple, yet powerful way. Like its single phase counterpart, multiphase percolation theory describes the connectivity of discrete inclusion assemblies as a function of phase topology. In addition, multiphase percolation employs basic laws to distinguish separate classes of objects and is characterized by its dependency on the order in which the different phases appear. This paper examines two applications of multiphase percolation theory: the first considers how the presence of bubble inclusions influences yield stress onset and growth in a magma’s crystal network; the second examines the effect of bimodal bubble-size distributions on magma permeability. We find that the presence of bubbles induces crystal clustering, thereby 1) reducing the percolation threshold, or critical crystal volume fraction, $\phi_c$, at which the crystals form a sample-spanning network, and 2) resulting in a larger yield stress for a given crystal volume fraction above $\phi_c$. This strengthening of the crystal network is also expected to occur when crystal clusters are formed due to processes other than bubble formation, such as heterogeneous crystallization, syneusis, and heterogeneity due to deformation or
flow. Further, we find that bimodal bubble size distributions can significantly affect
the permeability of the system beyond the percolation threshold. This study thus
demonstrates that larger scale structures and topologies can have significant effects
on macroscopic properties in multiphase materials.

Key words: magma, continuum percolation, multiphase materials, yield stress,
permeability

1 Introduction

Near the Earth’s surface, magma often exists as a multiphase material, typically comprising both gas bubbles and solid crystals embedded in a liquid
substrate (Figure 1). Quantifying magma permeability and related gas flow
rates through bubble networks, as well as the propagation of stresses through
crystal networks, are key to understanding a wide range of magmatic prop-
erties and processes. These include bubble and crystal content and related
magma rheology (Cashman and Blundy, 2000; Saar et al., 2001; Rust and
Manga, 2002; Melnik et al., 2005), emissions of volcanic gases (Edmonds et al.,
2003), coupled heat and volcanic gas transfer and related hydrothermal activ-
ity (Matsushima, 2003), focusing of magmatic-hydrothermal fluids (Candela,
1991), oxidation and cooling of pumice (Tait et al., 1998), welding (Sparks
et al., 1999), pressurization and destruction of conduit plugs and volcano flanks
(Melnik and Sparks, 2002), magma fragmentation in volcanic conduits (Klug
and Cashman, 1996) and subsequent expansion (Kaminski and Jaupart, 1997),
and transitions in eruption dynamics (e.g., Eichelberger et al., 1986) or a lack
thereof (Gonnermann and Manga, 2003).

The macroscopic rheological properties of magma are influenced by interac-
tions between embedded inclusions (e.g., Jerram et al., 2003; Noguchi et al., 2006). This is particularly evident when inclusions form clusters that span the substrate dimensions. The formation of space-spanning bubble networks, for example, coincides with the onset of magma permeability (Saar and Manga, 1999; Blower, 2001a). Such bubble networks may form open channels to the surface of the magma, preventing further bubble expansion (Gardner et al., 1996), causing the start of quenching (Rowland and Walker, 1990), and influencing eruption dynamics (Hammer et al., 1999; Clarke et al., 2007). At depth, the formation of a connected bubble-network may allow degassing into the surrounding rock, altering the melt rheology by inducing volatile exsolution and subsequent crystallization (Cashman and Blundy, 2000; Hammer and Rutherford, 2002). Crystals also have a significant effect on magma rheology. At low number-densities, crystals affect the magma viscosity as described by the Einstein-Roscoe equations (Einstein, 1906; Roscoe, 1953). The generation of space-spanning crystal networks marks a transition in magma rheology from Newtonian to Bingham-like fluid flow behavior (Lejeune and Richet, 1995; Philpotts et al., 1998; Saar et al., 2001).

While the complex thermodynamic, rheological, and stress field conditions during magma transport are often elusive, it is possible to examine the topological (i.e., geometric) conditions of object (e.g., particle, bubble) assemblies required for certain threshold material properties to occur, independent of the underlying processes causing these geometric conditions. Such geometric properties of embedded, space-spanning clusters of discrete inclusions are open to investigation by continuum percolation theory (Meester, 1996). Percolation theory is concerned with the connectedness of assemblies of objects. It is employed in a wide range of fields, including astrophysics, quantum me-
Fig. 1. a) Thin section image of a scoria sample from a cinder cone in the Cascades Mountains, Oregon, showing several connected bubbles (sub-spherical black), solid feldspar crystals (ragged black), and glassy rock matrix, i.e., liquid before rapid chilling (white). The dashed line indicates a sample-spanning bubble network. Crystals likely do not form a network in this particular sample. Modified from Saar (1998).

b) Example of the multiphase bubble and crystal simulations discussed in Section 3.

chanics, epidemiology, fire management, and traffic flow studies (Shante and Kirkpatrick, 1971; Meester, 1996). While percolation theory has also been applied to the study of magma rheology and permeability, much of this work has focused on single-phase percolation (e.g., Saar and Manga, 1999; Blower, 2001a; Saar et al., 2001; Rust and Cashman, 2004; Walsh and Saar, 2008).

Single-phase percolation deals only with objects of a particular type, for example either bubbles or crystals alone – ignoring codependent effects between separate embedded phases.

Here, we consider multiphase (or polychromatic, Zallen, 1977) continuum-percolation theory, which accounts for the separate natures of individual phases. As further discussed in Section 2, multiphase percolation differs from single-phase percolation in that it is path dependent: the structure of the inclusion clusters are a function of the order in which embedded objects appear within the material. This paper examines how path dependency influences magmatic yield stress and permeability, by considering two applications of multiphase percolation theory. The first, given in Section 3, examines the effects of bubble-
crystal codependencies on the development of magmatic yield stress. The sim-
ulations demonstrate that, in addition to any direct contribution to the yield
stress, bubbles also increase the strength of the magma by reducing the crys-
tal volume fraction required for crystal network formation and by rearranging
the resultant crystal network structure. The second example, presented in
Section 4, demonstrates how multiphase percolation theory can be applied to
separate inclusion types of the same phase by examining the permeability of
a bubble network with bimodally distributed radii. Although bimodal bubble
populations alter the volume fraction at the onset of permeability only slightly,
the distribution of bubble sizes strongly influences permeability development
beyond the percolation threshold.

2 Multiphase percolation

Continuum percolation is concerned with the connectedness of assemblies of
objects that are embedded within a continuous spatial domain (in contrast to
discrete percolation which deals with the connectedness of lattices of discrete
nodes). Of particular interest is the behavior of these assemblies at, and close
to, critical thresholds. One such threshold is the percolation threshold – the
moment at which objects in an assembly connect to form a space-spanning
cluster. This threshold is typically expressed in terms of a critical object vol-
ume fraction, $\phi_c$.

Most continuum percolation studies examine assemblies constructed from a
single population of inclusions. However, several materials – including many
magmas – contain multiple interacting embedded phases. Multiphase percola-
tion offers a simple way to model separate phases through the introduction of
Fig. 2. Exclusive assemblies of 100 circular and 100 rectangular inclusions: a) circular inclusions first; b) alternating circular and rectangular inclusions; and c) rectangular inclusions first.

basic interaction laws. Like single-phase percolation theory, multiphase percolation theory examines the behavior of particle clusters, but does so accounting for the differing relationships between the individual particle types.

A distinguishing feature of multiphase percolation is path dependency. With single-phase percolation, the assembly microstructure is unaffected by the order in which particles are added. In contrast, the order in which inclusions are added significantly influences the microstructure of assemblies with more than one phase. This is illustrated in Figure 2, which shows a two dimensional assembly constructed from separate populations of inclusions that are not allowed to overlap. If inclusions are added by alternating between the two phases, the space is shared equally by the two populations (Figure 2b). Conversely, if inclusions of one phase are added before the inclusions from the other, then most of the region will be occupied by the first phase, while the second phase is confined within the remaining space (Figures 2a and 2c).

When a second set of inclusions is added after an initial population of inclusions is present, the distribution of the second set of inclusions is described by a non-homogeneous Poisson process (Meester, 1996), such that the probability that the number of inclusions, $N$, within a volume, $V$, or area in two
dimensions, equals some number, $k$, is

$$P(N(V) = k) = \exp \left[ -n' \int_{\Omega} \int_{V} \Lambda \Phi \, d\mathbf{x} d\Theta \right] \left( n' \int_{\Omega} \int_{V} \Lambda \Phi \, d\mathbf{x} d\Theta \right)^k \frac{1}{k!}, \quad (1)$$

where $n'$ is defined as the local number density, i.e., the number of particles per unit area available to the second phase, and $\Phi = \Phi(\Theta)$ is a probability density distribution function describing the probability of finding a particle with a given orientation, $\Theta$. The function $\Lambda = \Lambda(\mathbf{x}, \Theta)$ is defined such that $\Lambda = 1$ if a particle from the second phase with an orientation $\Theta$ can be placed at position $\mathbf{x}$ without overlapping the first phase, and $\Lambda = 0$ otherwise. For large $V$,

$$\int_{\Omega} \int_{V} \Lambda \Phi \, d\mathbf{x} d\Theta = V(1 - \phi_{exab}), \quad (2)$$

where $\phi_{exab}$ is the excluded volume fraction occupied by the excluded volume, $v_{exab}$, of the region about object $a$ in which the center of object $b$ cannot be placed without causing overlap of the objects (Figure 3).

An average excluded volume fraction, $\langle \phi_{exab} \rangle$, is then given by the average excluded volume, $\langle v_{exab} \rangle$, obtained by averaging $v_{exab}$ over all possible relative orientations between objects $a$ and $b$. However, if at least one of the two objects is a circle (in two-dimensional systems) or a sphere (in three-dimensional systems), then $\langle v_{exab} \rangle = v_{exab}$ and hence $\langle \phi_{exab} \rangle = \phi_{exab}$, because spheres are orientation-independent. Thus for systems in which neither phase $a$ nor phase $b$ are comprised of spheres, all $v_{exab}$ and $\phi_{exab}$ discussed in this paper would be replaced by $\langle v_{exab} \rangle$ and $\langle \phi_{exab} \rangle$, respectively.

Assuming the first phase is itself constructed from a Poisson process, $\phi_{exab}$ is
expressed in terms of $v_{ex\,ab}$ as

$$\phi_{ex\,ab} = 1 - \exp[-n_a v_{ex\,ab}], \quad (3)$$

where $n_a$ is the particle number density of the first phase. Note that although the excluded volume of one particle with respect to the other is dependent only on the choice of particles, the same is not true for the excluded volume fractions of the phases, \textit{i.e.,}

$$v_{ex\,ab} = v_{ex\,ba}, \quad \text{but}$$
$$\phi_{ex\,ab} \neq \phi_{ex\,ba}. \quad (4)$$

The macroscopic particle density, $n$, of any phase is defined as

$$n = \lim_{V \rightarrow \infty} \sum_k k P(N(V) = k)/V, \quad (6)$$

giving the following relationship between the local and macroscopic particle number densities for phase $a$:

$$n'_a = n_a / (1 - \phi_{ex\,ab}). \quad (7)$$

Two examples of multiphase percolation are considered in the following sections. The first example investigates the effect of interacting bubble and particle inclusions on the onset and growth of yield stress. In the second example, two separate populations of spherical particles are used to model fluid flow through a network of bubbles with bimodal size distributions.
Fig. 3. Two-dimensional illustration of a) the excluded volume, $v_{ex_{ab}}$, of two objects a and b is the region about the object a within which the center of the second object b cannot be placed without causing overlap of the objects. b) The excluded volume of object a with respect to object b equals the excluded volume of object b with respect to object a, i.e., $v_{ex_{ab}} = v_{ex_{ba}}$.

3 Bubble and crystal networks

Nucleation and growth of bubbles in magma causes crystallization in response to the exsolution of volatiles from the melt (Cashman and Blundy, 2000). The crystals in magma in turn influence bubble development: by lowering the critical supersaturation pressure required for bubble nucleation (Gonnermann and Manga, 2007); and by altering the effective melt viscosity (Pinkerton and Stevenson, 1992).

In the absence of bubbles, the onset of a finite yield stress marks a transition in magma rheology from Newtonian to Bingham flow associated with the formation of a space-spanning crystal network (Lejeune and Richet, 1995; Philpotts et al., 1998). Elsewhere, this transition has been modeled as the single-phase percolation of assemblies of interpenetrating cuboids (Saar et al., 2001; Saar and Manga, 2002; Baker et al., 2002; Walsh and Saar, 2008). The particles in these models are allowed to interpenetrate fully, a so-called “soft core” model of contact, that simulates particle intergrowth in a quasi-static environment,
assumed for simplicity and first-order insight (Saar et al., 2001). More complex contact laws (e.g., so-called cherry-pit models) as well as physical or chemical processes may be applied as well, but for now, we focus on purely topological (geometrical) phase relationships, independent of the processes that caused them, in order to evaluate the effects of these relationships on magma rheology and permeability.

The current model simulates the microstructure of bubbles and crystals in magma with two populations of soft-core particles: spherical particles that represent bubbles; and cuboid particles that represent a network of intergrowing crystals as discussed in Saar et al. (2001). Both sets of particles are placed at random locations and orientations inside the simulated domain, approximating a zero-shear environment as may be expected in the center of plug flow. The spherical bubbles are placed first, followed by the cuboid crystals (Figure 1b), i.e., the three-dimensional equivalent of Figure 2a, meant to resemble the scoria sample with large subspherical bubbles and small crystals shown in Figure 1a. In contrast to that particular physical sample, crystal volume fractions in simulations can be increased to, and beyond, their percolation threshold. Similarly, bubble volume fractions can be varied numerically. If a crystal is found to overlap one or more of the bubbles then its position and orientation are reselected randomly until a non-overlapping configuration is found.

In a single-phase assembly at the percolation threshold, the space-spanning cluster of connected particles propagates densely throughout the assembly domain. Thus, a good approximation for the percolation threshold in the multiphase assembly is obtained by assuming that percolation occurs when the
Fig. 4. a) Crystal percolation threshold in the presence of bubbles, \( \phi_{cb,a} \), versus bubble volume fraction, \( \phi_a \). Simulated results are given by the circles, the approximate solution, given in Equation (8), is represented by the solid line. Bubble radii are one tenth the domain width, the particles are randomly oriented cuboids with dimensions (0.02, 0.004, 0.004) relative to the domain width. Periodic boundary conditions are enforced. The dashed curve reflects an expanding magma with a fixed ratio of crystal volume fractions, \( \phi_b \), to interstitial melt volume, \( 1 - \phi_a \), as further discussed in the main text. b) Minimum bubble separation that must be traversed by a percolating cluster \( w_{min} \) as a function of bubble volume fraction \( \phi_a \). local volume fraction, \( 1 - \phi_{ex,ab} \), reaches the percolation threshold, i.e., when

\[
\phi_{cb,a} \approx \phi_{cb} (1 - \phi_{ex,ab}),
\]

where \( \phi_{cb} \) is the single-phase percolation threshold for the crystal assembly. Note that the local volume fraction available for crystal network formation is not given by simply subtracting the bubble volume fraction from the system volume fraction. Rather, the local volume fraction is derived by subtracting the excluded volume fraction for crystal placements around bubbles, \( \phi_{ex,ab} \), from the system volume fraction. The value of \( \phi_{ex,ab} \) is found from Equation (3), where \( v_{ex,ab} \) is given by

\[
v_{ex,ab} = \frac{4\pi}{3} R^3 + \pi R^2 (l_1 + l_2 + l_3) + 2R(l_1l_2 + l_1l_3 + l_2l_3) + l_1l_2l_3,
\]

in which \( l_i \) are the dimensions of the cuboid crystal particles and \( R \) is the bubble radius. The theoretical (solid line in Figure 4a) and simulated perco-
lation thresholds (circles in Figure 4a) are in good agreement for low bubble-volume-fractions, however, the two diverge at higher bubble volume fractions. Equation 8 accounts for the volume occupied by the bubble phase, but not the effect of the bubbles on the crystals’ ability to form a percolating cluster. At small bubble volume fractions the crystals’ ability to form a connected network is relatively unimpeded. However, at higher bubble volume fractions, the diminished space between bubbles has a greater impact on the connected crystal network. The sketch in Figure 4b illustrates that for a given bubble number density, \( n \), there is a critical bubble radius, \( R_c \), at which the interstitial melt surrounding the bubbles will itself cease to form a connected network from one side of the assembly to the other. This critical bubble radius is given by

\[
R_c = -\left[ \frac{3}{4\pi n} \ln(1 - \phi_{cm}) \right]^{1/3},
\]

where \( \phi_{cm} \) is the critical percolation threshold for the melt phase with respect to interpenetrating spherical bubbles (\( \phi_{cm} \approx 0.97 \), Kertesz, 1981; Elam et al., 1984). This implies that the crystal cluster must traverse a gap between two bubbles separated by a minimum width

\[
w_{\text{min}} = 2(R_c - R) = 2R \left( \left[ \frac{\ln(1 - \phi_{cm})}{\ln(1 - \phi_a)} \right]^{1/3} - 1 \right),
\]

for \( R_c > R \), to form a percolating network. Plotting \( w_{\text{min}} \) as a function of the bubble volume fraction (Figure 4b), we see that the simulated and estimated values of the excluded volume begin to diverge as \( w_{\text{min}} \) falls below 10 times the maximum crystal dimension, approximately the same point at which finite size effects are observed in single phase percolation simulations (Saar and Manga, 2002).
Figure 4a is somewhat analogous to a phase diagram: the percolation threshold delineates suspensions that behave as Newtonian fluids, from those that behave as Bingham liquids; paths within this plot represent families of assemblies that share particular characteristics. For example, assemblies with a fixed ratio, $\lambda$, of crystal volume fraction, $\phi_b$, to interstitial melt volume, $1 - \phi_a$, are given by curves of the form:

$$\phi_b = \lambda (1 - \phi_a).$$

(12)

Equation (12) represents a depressurizing, and thus expanding, magma, where changes in liquid mass and volume are negligible while volatiles exsolve, forming expanding gas bubbles of increasing volume fraction. For a range of $\lambda$ values, these curves (e.g., dashed curve in Figure 4) pass through the critical threshold, even without further crystallization. Along these curves, increasing the number of bubbles in the suspension has the effect of increasing the strength of the assembly. This is due to the rearrangement of crystals within the suspension and is in addition to any direct contribution the bubbles may make to the onset and growth of yield stress (e.g., Ryerson et al., 1988; Gardiner et al., 1998), which is not accounted for by this model.

Any additional crystallization, due to cooling or degassing once a sample-spanning bubble network is formed, would shift the dashed curve in Figure 4 upwards to higher crystal volume fractions, $\phi_b$, facilitating the transition from Newtonian fluid to Bingham liquid. The combination of crystal and bubble nucleation and growth results in paths from the lower left toward the upper right of Figure 4 where the slopes of those paths depend on the crystallization and volatile exsolution behavior. Therefore, the exact onset of yield stress will depend on the crystallization and volatile exsolution path and on the location
of the percolation threshold curve, where the latter is itself dependent on crystal and bubble volume fractions, but also on crystal and bubble shapes, orientations, and size distributions. The effect of the latter is further discussed in Section 4.

The formation of a macroscopic, but fragile, crystal network at the percolation threshold, discussed so far, marks the onset of a minimum yield stress, \( \sigma_{\text{yield}} \rightarrow 0 \) Pa (Saar et al., 2001). Walsh and Saar (2008) investigate how \( \sigma_{\text{yield}} \) may grow above the percolation threshold for various crystal topologies in the absence of bubbles. In the following, we discuss the effect of bubble-induced crystal rearrangement on yield stress growth above the percolation threshold. Yield stress growth is investigated with a crystal network model that simulates the mechanical properties of the assembly. In this model, described in greater detail in Walsh and Saar (2008), the strength of particle bonds is determined by the amount of overlap between particles. Each bond is also assigned a failure criterion that controls the strain at which the bond is fractured.

Under single-phase percolation, the simulated yield stress, \( \sigma_{\text{yield}} \), due to phase \( b \) experiences power-law growth for volume fractions, \( \phi_b \), above phase \( b \)'s percolation threshold, \( \phi_{c_b} \) (Walsh and Saar, 2008), i.e.,

\[
\sigma_{\text{yield}} \propto (\phi_b - \phi_{c_b})^\nu, \tag{13}
\]

for \( \phi_b > \phi_{c_b} \). During multiphase percolation, however, a preexisting phase \( a \) (here bubbles) reduces phase \( b \)'s multiphase percolation threshold, \( \phi_{c_{ba}} \), compared to the single phase (\( b \)) percolation threshold, \( \phi_{c_b} \). This can be seen by replacing \( \phi_{c_b} \) with \( \phi_{c_{ba}} \) and combining Equation (13) with the earlier expres-
sion for the multiphase percolation threshold, given in Equation (8), yielding

$$\sigma_{\text{yield}} \propto [\phi_b - \phi_c(1 - \phi_{exab})]^{\nu},$$  \hspace{1cm} (14)

for $\phi_b > \phi_c(1 - \phi_{exab})$. As required Equation (14) reduces to Equation (13) when the bubble phase, $a$, is not present ($\phi_{exab} = 0$).

The value of the power-law exponent, $\nu$, is roughly constant in single phase assemblies ($\sim 3.5$), with little variation ($\sim \pm 0.2$) in response to changes to particle alignment and particle shape aspect ratio for both oblate and prolate particle assemblies (Walsh and Saar, 2008). However, this is not true for the multiphase assemblies studied here. As the bubble volume fraction is increased from $\phi_a = 0.0$ to $\phi_a = 0.5$, the measured value of the exponent increases from 3.6 to 4.6 (Figure 5).

The earlier onset of the percolating crystal network and the increase in the power-law exponent observed in the multiphase assembly are due to the heterogeneous spatial distribution of the crystals caused by the bubbles’ presence. Both effects increase the crystal network strength for a given crystal volume fraction. This is at least as important as other microstructural characteristics such as particle shape and orientation, particularly in determining the growth of yield stress with increasing crystal volume fractions. Heterogeneity in the crystal network is also a factor in determining the yield stress of single phase systems. Crystals tend to cluster in systems without bubbles as a result of heterogeneous crystallization (Hoover et al., 2001; Jerram et al., 2003) and syneusis (clustering due to hydrodynamic interactions) as discussed by Schwindinger and Anderson (1989). Multiphase percolation may also be employed in simulating such systems, as illustrated in the following section.
Fig. 5. Log-log plot of the crystal assembly yield stress (normalized by the maximum simulated yield stress) versus $\phi - \phi_c (1 - \phi_{exab})$ for different bubble volume-fractions, $\phi_a$, above the percolation threshold. As shown in the inset, the resultant power-law exponent, $\nu$, is not constant, but instead varies between 3.6 and 4.6 for the investigated values of $\phi_a$.

4 Bimodal bubble networks

Multiphase percolation analysis is not confined to assemblies of different physical phases. It is also applicable to problems involving distinct populations of inclusions from the same phase. Here this is illustrated by considering gas flow through bubble assemblies with bimodal size distributions.

Magmas display both exponential (Mangan and Cashman, 1996) and power-law bubble-size distributions (Klug et al., 2002; Namiki et al., 2003). Exponential bubble-size distributions are attributed to bubble development under steady-state conditions (Marsh, 1988), while power-law distributions may be due to bubble coalescence (Gaonac’h et al., 1996) or successive waves of nucleation (Blower et al., 2002). The latter scenario is considered in this section, where we simulate the simplest case in which two separate waves of bubble nucleations occur (Figure 6). However, while details may vary, our general conclusions, regarding the effects of differing bubble sizes on permeability, should be applicable to any system with non-uniform bubble sizes.
The two waves of nucleation are modeled by two populations of spherical bubbles. The centers of the first population are chosen at random, the centers of the second are also selected randomly with the condition that they do not lie within a given radial distance from the centers of the first population. The excluded region about the centers of the first population of bubbles represents the size of the first set of particles at the moment when the second set nucleates. Once the centers of the second set of bubbles have been selected, both sets of bubbles are increased to their final sizes and allowed to interpenetrate. As before with soft-core crystals, interpenetrating bubbles are a simplification that does not account for physical or chemical processes causing for example bubble resistance or tendency to coalesce or drain inter-bubble films. However, particularly for high viscosity, rhyolitic melts, as well as for rapidly chilled scoria formed during fountain eruptions of low viscosity basalts (Figure 1a), this topological approach allows some fundamental insights into geometric requirements, necessary for the formation of bubble networks and related finite system permeabilities.

Figure 7 shows the volume fractions of the two bubble phases at the percolation threshold. There is relatively little difference in the overall bubble-volume fraction at which percolation occurs, despite the bimodal size distribution. Similar results have also been observed elsewhere (e.g., Lorenz et al., 1993; Saar and Manga, 2002), although this may not hold in bimodal assemblies with vastly different bubble radii, $R_L >> R_s$ (Phani and Dhar, 1984).

Although the percolation threshold is not altered dramatically, the bimodal distribution affects the assembly’s macroscopic permeability. We calculate the permeability of the simulated assemblies using a network model based on that given by Blower (2001b). In this model, the permeability of the system is
Fig. 6. Schematic two-dimensional illustration of two waves of bubble nucleation and growth: (a) the first wave of bubbles nucleate and (b) begin to grow; (c) a later round of nucleation is triggered with new bubbles that appear between the old; (d) followed by a second round of bubble expansion. Actual simulations contain from 1000 to 72000 three-dimensional bubbles and employ periodic boundary conditions.

Fig. 7. Percolation threshold, i.e., critical total bubble volume fraction, $\phi_c = \phi_s + \phi_L$, for assemblies of bimodally distributed spheres, where $\phi_s$ and $\phi_L$ are the small and large bubble volume fractions, respectively.

calculated by assigning a resistance to flow to each of the apertures between the bubbles according to an approximate expression given by Feng et al. (1987).

Fluid is transmitted through the percolating backbone of the assembly, i.e., the space-spanning bubble network, excluding dead-ends (Figure 8). The ratio of large bubbles to small bubbles in the percolating backbone differs from the
Fig. 8. Normalized pressure distribution through the percolating backbone, calculated based on the network model of Blower (2001b). The ratio of large to small bubbles in the backbone is approximately 1 to 3, whereas the ratio over the entire assembly is approximately 1 to 26. Arrows indicate bottleneck locations caused by smaller bubbles.

The ratio of bubbles in the assembly as a whole: the ratio of large to small bubbles in the backbone is approximately 1 to 3, whereas the ratio over the entire assembly is approximately 1 to 26. From this it might be deduced that flow through the percolating backbone is controlled by the larger bubbles. However, the permeability of the pore network is complicated by the presence of the smaller bubbles, which create bottlenecks that restrict flow.

Figure 9 shows simulated permeabilities of bimodal bubble assemblies with different number densities of small spherical bubbles of radius $R_s = 0.01L$ against the volume fraction of the large-bubble population of radius $R_L = 0.04L$, where $L$ is the side length of the simulation bounding box. The number of small bubbles in the simulations range between 0 and 70,000 in increments of 3500; the number of large bubbles range between 1000 and 2000 in increments of 100. Permeabilities for each combination of small and large bubble number densities are calculated based on the median value obtained from twenty simulations.
As Figure 9 demonstrates, at low volume fractions of large bubbles, the permeability of the assembly is mostly determined by the number density of the small bubbles. However, for large-sphere volume fractions of $\phi_L \gtrsim 0.3$, the number density of the small bubbles has a negligible effect on the overall permeability. The shift in behavior is most easily explained by critical path analysis (e.g., Ambegaokar et al., 1971; Pollak, 1972; Hunt, 2005). When the large-bubble volume fraction is insufficient to form a percolating network on its own, any fluid flow must necessarily pass through the small bubble population. Thus the flow runs in series through the large and small bubble populations, where permeability is dominated by the smaller apertures of the smaller bubbles. Once the volume fraction of larger bubbles reaches the percolation threshold for assemblies of spheres, $\phi_L = 0.2895$ (Rintoul and Torquato, 1997), there is an increase in the macroscopic permeability. At this point the flow runs in parallel through the large and small bubble networks and is thus dominated by the larger apertures between the large bubbles.

Comparisons with real pumice and vesicular basalt samples under different conditions show that single phase percolation can both over- and underestimate the onset and development of percolation in magma (Saar, 1998; Saar and Manga, 1999; Rust and Cashman, 2004). Higher permeability estimates have been attributed to the collapse of the vesicular network, resulting in more elongated pores and preferential flow paths (Saar, 1998; Saar and Manga, 1999). In simulations of fully interpenetrating poly-disperse bubble assemblies, Blower (2001a) also found that a distribution of bubble sizes resulted in an increase in the permeability. However, as predicted by Blower et al. (2001), it is theoretically possible to develop arbitrarily low-permeability/high-porosity bubble assemblies, if the new bubble positions are dependent on existing bub-
ble locations. The results of our multiphase simulation support that prediction. In the assemblies simulated with large volume fractions of large bubbles, the small-bubble population adds to the void space without contributing significantly to the overall permeability. Thus, the permeability of the multiphase assembly is less than that of a single-phase assembly of large bubbles with the same volume fraction.

The results of our multiphase simulations of bimodal bubble-size distributions show that reproducing the bubble-size distribution is not sufficient to correctly simulate magma permeability. Larger-scale heterogeneous structures in the bubble networks must also be accounted for. As with crystal assemblies discussed in Section 3, these heterogeneities may also arise as a result of processes other than the successive waves of nucleation discussed here. Bubble deformation as well as preferred bubble orientations and spatial distributions, for example, occur in response to shearing (Rust and Manga, 2002; Tucker and Moldenaers, 2002; Quilliet et al., 2005). Therefore, correctly characterizing these larger-scale structures in rock samples or deriving such structures in simulations of magma transport, is vital in determining the overall permeability of magmatic systems.

5 Conclusion

Multiphase percolation theory offers a simple way to simulate the effects of inclusion interactions on the macroscopic properties of magma. Here this is demonstrated by considering two cases of interest: 1) the effect of bubbles on the onset and growth of yield stress in crystal melt suspensions, and 2) the role of bimodal bubble size distributions on the permeability of bubble networks.
Fig. 9. Calculated dimensionless permeability of bimodal-bubble-size assemblies for different populations of large \((R_L = 0.04L)\) and small \((R_s = 0.01L)\) bubbles in a bounding box of side length, \(L\). Permeabilities are plotted against volume fraction of large bubbles, with each line representing assemblies with the same number density of small bubbles. Each data point is the median value from 20 simulations normalized by the square of the radius of the larger bubbles. Median, rather than mean, values are used as they show less variation in the calculated permeabilities, particularly at volume fractions close to the percolation threshold of the system. Permeability calculations are not given for some data points at small values below the percolation threshold of the large-bubble volume fraction, \(\phi_L < 0.2895\). At these data points, the calculated permeability is zero, as in these cases the combination of small and large spheres is not sufficient to form a percolating cluster.

The results from the simulations of multiphase bubble and crystal assemblies suggest that the presence of bubbles in a magma acts to increase its yield stress by rearranging the crystal network. This contribution is in addition to any increase in the overall strength of magma due to the presence of the bubble network itself. In these simulations, the excluded volume of the bubbles controls the onset of crystal percolation, rather than the bubble volume, as might be expected. Thus, increasing the bubble volume fraction lowers the crystal percolation threshold for a given ratio of crystal to melt volume. As in studies of single-phase percolation, the yield stress is shown to have a power-law growth for crystal volume-fractions above the crystal percolation threshold. However, as the bubble phase is increased the exponent of the power-law is larger than that measured in single-phase simulations, and increases with in-
creasing bubble volume fractions. Therefore, clustering or clumping of crystals, by whatever mechanism, may be a significant contributor to magmatic yield stress both by lowering the percolation threshold and increasing subsequent yield stress growth.

While different ratios of large and small bubbles in the simulated bimodal bubble assemblies do not dramatically alter the percolation threshold, the effect on the permeability of the assembly is significant. Although larger bubbles are overrepresented in the percolating cluster, it is the smaller bubbles that determine the overall permeability close to the percolation threshold. Before the larger bubble population has reached its percolation threshold, any percolating pathway must necessarily pass through the set of smaller bubbles, resulting in bottlenecks that lower the permeability. There is an increase in the permeability in assemblies with larger number densities of large bubbles due to the formation of a percolating cluster in the larger bubbles. When this occurs, flow through the assembly bypasses the smaller-bubble network. Through this mechanism, assemblies of polydisperse bubble sizes exhibit lower permeabilities than predicted by single-phase percolation theory.

In both sets of simulations, the multiple interdependent phases considered introduce larger-scale heterogeneities absent in single-phase assemblies. Multiphase percolation is just one mechanism whereby such larger-scale structures might arise; other mechanisms include syneusis, heterogeneous crystallization and bubble nucleation, as well as heterogeneities arising from deformation and flow. Independent of the underlying cause, this study demonstrates that such larger scale structures can have a significant role in determining the macroscopic properties of magmas and other multiphase materials. As a result, it may be critical to consider multiple coexisting phases and their topologies.
(e.g., shape and size distribution, clustering, and connectivity), \textit{i.e.}, not just the mass or volume fractions of phases, when studying magma flow and degassing as well as other multiphase processes.

6 Acknowledgments

MOS thanks the George and Orpha Gibson endowment for its generous support of the Hydrogeology and Geoﬂuids research group. In addition, this material is based upon support by the National Science Foundation (NSF) under Grant No. EAR-0510723 and Grant No. DMS-0724560. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF. We also gratefully acknowledge the use of resources from the University of Minnesota Supercomputing Institute (MSI).

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